

The Hölder Inequality for KMS States

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Abstract

We prove a Hölder inequality for KMS States, which generalises a well-known trace-inequality. Our results are based on the theory of non-commutative L_p -spaces.

1 Introduction

Trace inequalities have played a key role both in mathematics and quantum statistical mechanics [4, 12, 13]. In recent years numerous trace inequalities have been generalised to σ -finite von Neumann algebras, for example the Golden-Thompson and Peierls-Bogolubov inequalities [2]. In this short note, we generalize the Hölder trace-inequality. The latter has been used, for example, by Ruelle to construct interacting Gibbs states [21][22] in a box and then control their thermodynamic limit. While trace inequalities are useful for quantum systems constrained to a finite volume, there are good reasons to abandon the boxes and study quantum statistical systems directly in infinite volume. As the generator of the time evolution will no longer have discrete spectrum, trace inequalities can not be applied. Thus the Hölder trace-inequality has to be replaced by the generalised inequality presented in Section 2. It was pointed out by Fröhlich [7] that the Hölder inequality given in Section 2 also plays a crucial role in the context of thermal quantum field theory.

The paper is organised as follows. In Section 2 we recall some basic notions of Tomita-Takesaki theory and state the main result. Section 3 contains an introduction to non-commutative L_p -spaces. Section 4 provides the proof of the main theorem.

2 The Main Result

In quantum statistical mechanics, thermal equilibrium states are characterised by the *KMS condition* [8], which is (a) a generalisation of the Gibbs condition

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to systems in infinite volume; (b) formulated in terms of analyticity properties of the correlation functions; and (c) can be derived from first principles, like *passivity* [20] or *stability* under small adiabatic perturbations of the dynamics [9].

Definition 2.1. Let \mathcal{A} be a C^* -algebra and $\{\tau_t\}_{t \in \mathbb{R}}$ be a strongly continuous group of $*$ -automorphisms of \mathcal{A} . A normalised positive linear functional ω_β over \mathcal{A} is called a (τ, β) -KMS state for the inverse temperature $\beta > 0$, if for all $A, B \in \mathcal{A}$ there exists a function $F_{A,B}$, which is continuous and bounded in the strip $0 \leq \Im z \leq \beta$ and analytic in the open strip $0 < \Im z < \beta$, with boundary values given by

$$F_{A,B}(t) = \omega_\beta(A\tau_t(B)) \quad (1)$$

and $F_{A,B}(t + i\beta) = \omega_\beta(\tau_t(B)A)$ for all $t \in \mathbb{R}$.

The KMS-condition implies that ω_β is invariant under τ and therefore the latter can be unitarily implemented in the GNS representation $(\pi, \mathcal{H}, \Omega)$ associated to the pair $(\mathcal{A}, \omega_\beta)$. Weak continuity of τ ensures the existence of a generator L , called the *Liouvillean*, such that $\pi(\tau_t(A))\Omega = e^{-itL}\pi(A)\Omega$ and $L\Omega = 0$.

As the vector Ω is cyclic and separating for the von Neumann algebra $\mathcal{M} \doteq \pi(\mathcal{A})''$, the algebraic operations on \mathcal{M} define maps on the dense set $\mathcal{M}\Omega \subset \mathcal{H}$. Tomita's idea to study the $*$ -operation on \mathcal{M} turned out to be especially fruitful. It leads to an anti-linear operator S_\circ ,

$$S_\circ: A\Omega \mapsto A^*\Omega, \quad A \in \mathcal{M},$$

which is closable, and thus allows a polar decomposition for the closure $S = J\Delta^{1/2}$. The anti-linear involution J is called the *modular conjugation* and the positive albeit in general unbounded operator Δ is called the *modular operator*. The modular conjugation J satisfies $J^* = J$ and $J^2 = \mathbb{1}$, and induces a $*$ -anti-isomorphism $j: A \mapsto JA^*J$ between the algebra \mathcal{M} and its commutant \mathcal{M}' (Tomita's theorem).

More generally, an arbitrary normal faithful state over a von Neumann algebra \mathcal{M} is a $(\sigma, -1)$ -KMS state with respect to the modular automorphisms σ given by $A \mapsto \Delta^{is}A\Delta^{-is}$, $A \in \mathcal{M}$, $s \in \mathbb{R}$, at temperature $\beta = -1$ (see, e.g., [5]). To be precise, the strong continuity assumption, which is part of Definition 2.1, holds on the restricted C^* -dynamical system [23, Proposition 1.18] associated to the W^* -dynamical system (\mathcal{M}, σ) . Uniqueness of the modular automorphism ensures that $\Delta^{1/2} = e^{-\beta L/2}$.

The *standard positive cone* $\mathcal{P}^\sharp \subset \mathcal{H}$ is defined as

$$\mathcal{P}^\sharp \doteq \overline{\{JAJA\Omega : A \in \mathcal{M}\}} = \overline{\{\Delta^{1/4}A\Omega : A \in \mathcal{M}^+\}},$$

where the bar denotes norm closure [1]. Consequently, a KMS state on a C^* -dynamical system (\mathcal{A}, τ) gives rise to a von Neumann algebra in *standard form*, namely a quadruple $(\mathcal{H}, \mathcal{M}, J, \mathcal{P}^\sharp)$, where \mathcal{H} is a Hilbert space, \mathcal{M} is a von Neumann algebra, J is an anti-unitary involution on \mathcal{H} and \mathcal{P}^\sharp is a self-dual cone in \mathcal{H} such that:

- (i) $J\mathcal{M}J = \mathcal{M}'$;
- (ii) $JAJ = A^*$ for A in the center of \mathcal{M} ;
- (iii) $J\Psi = \Psi$ for $\Psi \in \mathcal{P}^\sharp$;
- (iv) $AJAP^\sharp \subset \mathcal{P}^\sharp$ for $A \in \mathcal{M}$.

The vector state induced by Ω extends the KMS state ω_β from \mathcal{A} to \mathcal{M} , and we denote this state by the same letter. Now set, for $p \in \mathbb{N}$ and $A \in \mathcal{M}^+$,

$$\|A\|_p \doteq \omega_\beta \left(\underbrace{e^{itL/p} A \cdots e^{itL/p} A}_{p\text{-times}} \right)_{|t=i\beta}^{1/p}. \quad (2)$$

The subscript indicates the analytic continuation of the map $t \mapsto F(t) \doteq \omega_\beta(e^{itL/p} A \cdots e^{itL/p} A)$ to $F(i\beta)$. To simplify the notation we will denote $F(i\beta)$ by $\omega_\beta(e^{-\beta L/p} A \cdots e^{-\beta L/p} A)$.

Theorem 2.2 (Hölder inequality). *Consider a (τ, β) -KMS state ω_β over a C^* -dynamical system (\mathcal{A}, τ) . Let $(z_1, \dots, z_n) \in \mathbb{C}^n$ be such, that $0 \leq \Re z_j$, $\sum_{j=1}^n \Re z_j \leq 1$, and let p_j be the smallest, positive even integer such that $\frac{1}{p_j} \leq \min\{\Re z_{j+1}, \Re z_j\}$, with $z_{n+1} = z_n$ and $z_0 = z_1$. Then*

$$\left| \omega_\beta(A_n e^{-z_n \beta L} \cdots A_1 e^{-z_1 \beta L} A_0) \right| \leq \|A_0\|_{p_0} \cdots \|A_n\|_{p_n} \quad (3)$$

for all $A_0, \dots, A_n \in \mathcal{M}^+$.

Remarks

- (i) Although the multi-boundary Poisson kernels [23, Lemma 4.4.8] for the domain $I^{(n)}$ can be computed explicitly (the computation can be traced back to Widder [27]), it seems unlikely that the Hölder inequality (3) can be derived using only methods of complex analysis (unless $n = 2$).
- (ii) Let \mathcal{M}_0 denote a weakly dense sub-algebra of analytic¹ elements in \mathcal{M} . It follows that, for $p \in \mathbb{N}$ and $A \in \mathcal{M}_0^+$,

$$\|A\|_p = \omega_\beta(\tau_{i\beta/2p}(A) \cdots \tau_{i(2p-1)\beta/2p}(A) \tau_{i\beta}(A))^{1/p}. \quad (4)$$

Thus Theorem 2.2 is a generalisation of the Hölder inequality for Gibbs states, as stated, for example, in [16, 17].

Two more aspects of Theorem 2.2 are notable. Firstly, it estimates a non-commutative expression in terms of essentially commutative ones, which can be evaluated using spectral theory, and secondly, the bounds are uniform in $\Im z_j$,

¹An element $A \in \mathcal{M}$ is called analytic for τ_t if there exists a strip $I_\lambda = \{z \in \mathbb{C} : |\Im z| < \lambda\}$ in \mathbb{C} , and a function $f: I_\lambda \rightarrow \mathcal{M}$, such that (i) $f(t) = \tau_t(A)$ for $t \in \mathbb{R}$, and (ii) $z \mapsto \phi(f(z))$ is analytic for all $\phi \in \mathcal{M}_*$.

$j = 1, \dots, n$. The proof of Theorem 2.2 relies on the theory of non-commutative L^p -spaces, but the appeal of the theorem may well be that knowledge of non-commutative integration theory is not required in order *to apply* the inequality.

In quantum statistical mechanics the uniformity in imaginary time is useful for establishing the existence of real time Greens functions from the Schwinger functions. Beyond quantum statistical mechanics, inequality (3) is also useful in constructive quantum field theory. In [7] Fröhlich argued that the Hölder inequality will guarantee the existence of thermal Wightman functions for a certain class of models. A complete proof of this claim is given in [14]. Additionally, in a forthcoming work by M. Rouleux and the first author, the Hölder inequality is used to show that the Wightman distributions of the $P(\phi)_2$ model on the de Sitter space satisfy a micro-local spectrum condition.

3 Non-commutative L_p -spaces

Normal states over a von Neumann algebras provide a non-commutative extension of classical integration theory, i.e., commutative L^p -spaces, and one recovers the latter in case the algebra is abelian [18]. Among the many approaches to non-commutative L_p -spaces [6, 10, 11, 15, 19, 24, 26], Araki and Masuda's approach [3] is best suited for our purposes. We start with a short introduction to relative modular operators for weights. A more elaborate discussion of relative modular operators can be found in [25].

3.1 Relative Modular Operators

Consider a general (σ -finite) von Neumann algebra \mathcal{M} and let ϕ be a normal semi-finite weight on \mathcal{M} . The semi-cyclic representation² [25] makes it possible to define an anti-linear operator $S_{\phi, \Omega}$ by

$$S_{\phi, \Omega} A \Omega \doteq \xi_{\phi}(A^*) , \quad A \in \mathcal{N}_{\phi}^* ,$$

where $\mathcal{N}_{\phi} \doteq \{A \in \mathcal{M} : \phi(A^* A) < \infty\}$, and $\xi_{\phi}(A)$ is the semi-cyclic representation of $A \in \mathcal{N}_{\phi}$ in

$$\mathcal{H}_{\phi} \doteq \overline{\mathcal{N}_{\phi} / \ker \phi} .$$

$S_{\phi, \Omega}$ is closable and the closure $\overline{S_{\phi, \Omega}}$ has a polar decomposition $\overline{S_{\phi, \Omega}} \doteq J_{\phi, \Omega} \Delta_{\phi, \Omega}^{1/2}$. It is noteworthy that

$$\Delta_{\phi, \Omega} = S_{\phi, \Omega}^* \overline{S_{\phi, \Omega}} ,$$

is a positive, in general unbounded, operator on the *original* Hilbert space \mathcal{H} . If ϕ is a vector state associated to $\xi \in \mathcal{H}$ such that $\phi(x) = (\xi, x\xi)$, then $\xi_{\phi}(A) = A\xi$ and we denote $\Delta_{\phi, \Omega}$ by $\Delta_{\xi, \Omega}$ and $J_{\phi, \Omega}$ by $J_{\xi, \Omega}$. In order to keep the notation simple, $e^{-\beta L/2}$ will from now on be written as $\Delta^{1/2} \equiv \Delta_{\Omega, \Omega}^{1/2}$.

²The semi-cyclic representation is a generalisation of the GNS representation to weights.

A key role in the proof of Theorem 2.2 will be played by the following estimate: define, for any $\alpha > 0$, a set

$$I_\alpha^{(n)} \doteq \{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n \Re z_j \leq \alpha, 0 \leq \Re z_j\}. \quad (5)$$

Let $z \in I^{(n)} \equiv I_1^{(n)}$ and $z'_m, z''_m \in \mathbb{C}$ be such that $\Re z'_m, \Re z''_m > 0$, $z'_m + z''_m = z_m$ and

$$\Re z_1 + \dots + \Re z_{m-1} + \Re z''_m \leq 1/2, \quad (6)$$

$$\Re z_n + \dots + \Re z_{m+1} + \Re z'_m \leq 1/2. \quad (7)$$

Under these conditions, Araki [3, Lemma A] has shown³ that for $\phi_1, \dots, \phi_n \in \mathcal{M}_*^+$ and $X_0, \dots, X_n \in \mathcal{M}$

$$\begin{aligned} & \left| \left(\Delta_{\phi_m, \Omega}^{\bar{z}'_m} X_m^* \Delta_{\phi_{m+1}, \Omega}^{\bar{z}_{m+1}} \dots \Delta_{\phi_n, \Omega}^{\bar{z}_n} X_n^* \Omega, \Delta_{\phi_m, \Omega}^{\bar{z}''_m} X_{m-1} \Delta_{\phi_{m-1}, \Omega}^{z_{j-1}} \dots \Delta_{\phi_1, \Omega}^{z_1} X_0 \Omega \right) \right| \\ & \leq \left(\prod_{j=0}^n \|X_j\| \right) (\Omega, \mathbb{1} \Omega)^{z_0} \left(\prod_{j=1}^n \phi_j(\mathbb{1})^{\Re z_j} \right), \end{aligned} \quad (8)$$

with $z_0 = 1 - \sum_{j=1}^n \Re z_j$.

Remark 3.1. Consider the space of $n \times n$ -matrices $M_n(\mathbb{C}) \ni \xi, \eta$ equipped with the inner product $\langle \xi, \eta \rangle = \text{Tr } \xi^* \eta$ and two positive matrices $0 < \nu, \omega \in M_n(\mathbb{C})$. Moreover, assume that $\text{Tr } \omega = 1$. Now apply the Hölder trace inequality [16]

$$|\text{Tr } \omega A B| \leq \|A\|_{\omega, p} \|B\|_{\omega, q}, \quad p^{-1} + q^{-1} = 1, \quad (9)$$

where $\langle A \rangle_\omega \doteq \text{Tr } \omega A$ and $\|A\|_{\omega, p}^p \doteq \text{Tr } (\omega^{1/2p} |A| \omega^{1/2p})^p$, to the relative modular operator $\Delta_{\nu, \omega}$, which satisfies $\Delta_{\nu, \omega}^{1/p} \xi = \nu^{1/p} \xi \omega^{-1/p}$ for $p \in \mathbb{N}$. Thus, for $1/p + 1/q = 1$,

$$\begin{aligned} |\langle A_2 \Delta_{\nu_2, \omega}^{1/p} A_1 \Delta_{\nu_1, \omega}^{1/q} A_0 \rangle_\omega| & \leq \left(\prod_{j=0}^2 \|A_j\|_\infty \right) \|\Delta_{\nu_2, \omega}^{1/p}\|_{\omega, p} \|\Delta_{\nu_1, \omega}^{1/q}\|_{\omega, q} \\ & = \left(\prod_{j=0}^2 \|A_j\|_\infty \right) \langle \mathbb{1} \rangle_{\nu_2}^{1/p} \langle \mathbb{1} \rangle_{\nu_1}^{1/q}. \end{aligned} \quad (10)$$

3.2 Positive Cones and L_p -Spaces for von Neumann Algebras

Consider a general (σ -finite) von Neumann algebra \mathcal{M} in standard form with cyclic and separating vector Ω . For $2 \leq p \leq \infty$, Araki and Masuda define [3, Equ. (1.3), p. 340]

$$L_p(\mathcal{M}, \Omega) \doteq \left\{ \zeta \in \bigcap_{\xi \in \mathcal{H}} D(\Delta_{\xi, \Omega}^{\frac{1}{2} - \frac{1}{p}}) : \|\zeta\|_p < \infty \right\},$$

³Note that, in contrast to [3], our inner product is linear in the second entry.

where

$$\|\zeta\|_p = \sup_{\|\xi\|=1} \|\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}} \zeta\|.$$

For $1 \leq p < 2$, $L_p(\mathcal{M}, \Omega)$ is defined as the completion of \mathcal{H} with respect to the norm

$$\|\zeta\|_p = \inf\{\|\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}} \zeta\| : \|\xi\| = 1, s_{\mathcal{M}}(\xi) \geq s_{\mathcal{M}}(\zeta)\}.$$

Here $s_{\mathcal{M}}(\xi)$ denotes the smallest projection in \mathcal{M} , which leaves ξ invariant. The cones [3, Equ. (1.13)]

$$\mathcal{P}^\alpha \doteq \overline{\{\Delta^\alpha A \Omega : A \in \mathcal{M}^+\}}, \quad 0 \leq \alpha \leq 1/2,$$

can be used to define the positive part of $L_p(\mathcal{M}, \Omega)$ [3, Equ. (1.14), p. 341]:

$$L_p^+(\mathcal{M}, \Omega) \doteq L_p(\mathcal{M}, \Omega) \cap \mathcal{P}_\Omega^{1/(2p)}, \quad 2 \leq p \leq \infty. \quad (11)$$

Note that these are not operator spaces. The connection to the operator algebra \mathcal{M} is made through auxilliary spaces $\mathcal{L}_p(\mathcal{M}, \Omega)$, which consist of formal expressions $A = u\Delta_{\phi,\Omega}^{1/p}$ with $\phi \in \mathcal{M}_*^+$ and u a partial isometry satisfying $u^*u = s(\phi)$ (the support projection of ϕ). The set of formal products

$$X_0 \Delta_{\phi_1,\Omega}^{z_1} X_1 \cdots \Delta_{\phi_n,\Omega}^{z_n} X_n, \quad (12)$$

is denoted by $\mathcal{L}_p^*(\mathcal{M}, \Omega)$. Here is $X_j \in \mathcal{M}$ ($j = 0, \dots, n$), $\phi_j \in \mathcal{M}_*^+$ ($j = 1, \dots, n$) and $\vec{z} = (z_1, \dots, z_n) \in I_{1-(1/p)}^{(n)}$. On the subset $\mathcal{L}_{p,0}^*(\mathcal{M}, \Omega) \subset \mathcal{L}_p^*(\mathcal{M}, \Omega)$, characterized by the condition $\sum_{j=1}^n \Re z_j = 1 - (1/p)$, it is possible to implement the star operation. The adjoint of a generic element (12) in $\mathcal{L}_{p,0}^*(\mathcal{M}, \Omega)$ is defined to be

$$X_n^* \Delta_{\phi_n,\Omega}^{\bar{z}_n} \cdots X_1^* \Delta_{\phi_1,\Omega}^{\bar{z}_1} X_0^*. \quad (13)$$

A multiplication can be defined, using the product in \mathcal{M} to connect the formal expressions: $BC \in \mathcal{L}_{r,0}^*(\mathcal{M}, \Omega)$ for $B \in \mathcal{L}_{p,0}^*(\mathcal{M}, \Omega)$, $C \in \mathcal{L}_{q,0}^*(\mathcal{M}, \Omega)$ and $r^{-1} = p^{-1} + q^{-1} - 1$.

If $r^{-1} = \sum_{j=1}^n (p_j)^{-1}$, $r^{-1} + r^{-1} = 1$, $\xi_j \in L_{p_j}(\mathcal{M}, \Omega)$, $X_j \in \mathcal{M}$ ($j = 0, \dots, n$), and $\xi_j = u_j \phi_j^{1/p_j}$, ($j = 1, \dots, n$) is the polar decomposition, then the product

$$\xi = X_0 \xi_1 X_1 \xi_2 \cdots \xi_n X_n \in L_r(\mathcal{M}, \Omega) \quad (= L_{r'}(\mathcal{M}, \Omega)^*)$$

is defined by

$$\langle \xi, \xi' \rangle = \omega(\Delta_{\phi',\Omega}^{1/r'} u'^* X_0 u_1 \Delta_{\phi_1,\Omega}^{1/p_1} X_1 u_2 \Delta_{\phi_2,\Omega}^{1/p_2} \cdots u_n \Delta_{\phi_n,\Omega}^{1/p_n} X_n) \in L_r(\mathcal{M}, \Omega)$$

where $\xi' \in L_{r'}(\mathcal{M}, \Omega)$ and $\xi' = u' \phi'^{1/r'}$ is its polar decomposition.

Araki's inequality (8) now entails a Hölder inequality: let $\zeta_1 \in L_p(\mathcal{M}, \Omega)$ and $\zeta_2 \in L_{p'}(\mathcal{M}, \Omega)$ for $p^{-1} + p'^{-1} = r^{-1}$, then

$$\|\zeta_1 \zeta_2\|_r \leq \|\zeta_1\|_p \|\zeta_2\|_{p'}. \quad (14)$$

Thus the product $\zeta_1\zeta_2$ is in $L_r(\mathcal{M}, \Omega)$ and, as the case $p^{-1} + p'^{-1} = 1$ suggests, the topological dual $L_p(\mathcal{M}, \Omega)^*$ of $L_p(\mathcal{M}, \Omega)$ is $L_{p'}(\mathcal{M}, \Omega)$. For $A \in \mathcal{L}_p(\mathcal{M}, \Omega)$ and $B \in \mathcal{L}_p(\mathcal{M}, \Omega)^*$, the corresponding duality bracket is given by

$$\langle A, B \rangle = (A\Omega, B\Omega), \quad (15)$$

if Ω is in the domain of A and B . According to [3, Notation 2.3 (4)] A and B in $\mathcal{L}_p^*(\mathcal{M}, \Omega)$ are said to be equivalent, if (i) $1 \leq p \leq 2$ and $A\Omega = B\Omega$; (ii) if $2 \leq p \leq \infty$ and

$$\langle C, A \rangle = \langle C, B \rangle \quad (16)$$

for all C in $\mathcal{L}_p(\mathcal{M}, \Omega)$.

Another important property is, that for $1 \leq p \leq \infty$, $x \in \mathcal{M}$ and $\zeta \in L_p(\mathcal{M}, \Omega)$, the following inequality holds:

$$\|x\zeta\|_p \leq \|x\| \|\zeta\|_p. \quad (17)$$

It is evident from the definition of the L_p -spaces, that \mathcal{H} and $L_2(\mathcal{M}, \Omega)$ are equal. It is proven in [3] that $\mathcal{M} \cong L_\infty(\mathcal{M}, \Omega)$ as well as $\mathcal{M}_* \cong L_1(\mathcal{M}, \Omega)$.

4 Proof of the Main Result

Lemma 4.1. *Let $A_1, \dots, A_n \in \mathcal{M}^+$. Then there exist unique $\phi_j \in \mathcal{M}_*^+$ such that for $0 \leq p_j^{-1} \leq 1/2$*

$$\Delta_{\phi_j, \Omega}^{1/p_j} \Omega = \Delta^{1/2p_j} A_j \Omega \quad (j = 1, \dots, n) \quad (18)$$

and $\phi_j(\mathbb{1})^{1/p_j} = \|\Delta^{1/2p_j} A_j \Omega\|_{p_j}$. If also $\sum_{j=1}^n 1/p_j = 1/2$ holds, then

$$\Delta_{\phi_n, \Omega}^{1/p_n} \dots \Delta_{\phi_1, \Omega}^{1/p_1} \Omega = \Delta^{1/2p_n} A_n \Delta^{1/2p_n} \dots \Delta^{1/2p_1} A_1 \Omega \in \mathcal{H}. \quad (19)$$

Proof. Let $A_1, \dots, A_n \in \mathcal{M}^+$ and $0 \leq p_j^{-1} \leq 1/2$, $j = 1, \dots, n$. Then, by definition $\zeta_j \doteq \Delta^{1/2p_j} A_j \Omega \in \mathcal{P}_\Omega^{1/2p_j}$. An application of inequality (8) yields

$$\|\zeta_j\|_{p_j}^2 = \sup_{\|\xi\|=1} \|\Delta_{\xi, \Omega}^{(1/2)-(1/p_j)} \zeta_j\|^2 \quad (20)$$

$$= \sup_{\|\xi\|=1} \left(\Delta_{\xi, \Omega}^{(1/2)-(1/p_j)} \Delta^{1/2p_j} A_j \Omega, \Delta_{\xi, \Omega}^{(1/2)-(1/p_j)} \Delta^{1/2p_j} A_j \Omega \right) \quad (21)$$

$$\leq \sup_{\|\xi\|=1} (\xi, \mathbb{1}\xi)^{1-(2/p_j)} \omega(\mathbb{1})^{2/p_j} \|A_j\|^2 = \|A_j\|^2 < \infty, \quad (22)$$

which establishes, that $\zeta_j \in L_{p_j}(\mathcal{M}, \Omega)$. Thus, according to (11), $\zeta_j \in L_{p_j}^+(\mathcal{M}, \Omega)$. By [3, Theorem 3 (4), p. 342] there exists a unique $\phi_j \in \mathcal{M}_*^+$ such that $\zeta_j = \Delta_{\phi_j, \Omega}^{1/p_j} \Omega$ and $\phi_j(\mathbb{1})^{1/p_j} = \|\zeta_j\|_{p_j} = \|\Delta^{1/2p_j} A_j \Omega\|_{p_j}$.

Thus, by definition [3, Notation 2.3 (4)], $\Delta^{1/2p_j} A_j \Delta^{1/2p_j} \equiv \Delta_{\phi_j, \Omega}^{1/p_j}$ as elements in $\mathcal{L}_{p'_j, 0}^*(\mathcal{M}, \Omega)$, where $p_j^{-1} + p'_j{}^{-1} = 1$. Even though $\Delta_{\phi_j, \Omega}^{1/p_j}$ and $\Delta^{1/2p_j} A_j \Delta^{1/2p_j}$

may not be equal as operators, Lemma 7.7 (2) in [3] ensures, that their composition as elements of the spaces \mathcal{L}_p^* is well-defined: setting $B_1 = \Delta_{\phi_2, \Omega}^{1/p_2}$, $B_2 = -\Delta^{1/2p_2} A_2 \Delta^{1/2p_2}$ and $C_2 = \Delta_{\phi_1, \Omega}^{1/p_1}$, there holds $\sum_{i=1}^2 B_i = 0$ as elements in $L_{p_2}(\mathcal{M}, \Omega)$, and therefore, using the lemma cited,

$$\Delta_{\phi_2, \Omega}^{1/p_2} \Delta_{\phi_1, \Omega}^{1/p_1} \Omega \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta_{\phi_1, \Omega}^{1/p_1} \Omega \quad (23)$$

as elements in $L_{r_1}(\mathcal{M}, \Omega) = L_{r'_1}(\mathcal{M}, \Omega)^*$, where $r_1^{-1} + r'_1{}^{-1} = 1$, $r_1'^{-1} = p_1'^{-1} + p_2'^{-1} - 1$ and $1 \leq r'_1 \leq 2$ (in comparison to [3] indices and primed indices have swapped places). Note that this means $r_1^{-1} = p_2^{-1} + p_1^{-1}$. Using the same lemma once more (with the appropriate choices of C_2 and B_3, B_4) gives

$$\Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta_{\phi_1, \Omega}^{1/p_1} \Omega \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Omega \quad (24)$$

as elements in $L_{r'_1}(\mathcal{M}, \Omega)^*$. Together (23) and (24) imply

$$\Delta_{\phi_2, \Omega}^{1/p_2} \Delta_{\phi_1, \Omega}^{1/p_1} \Omega \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Omega \quad (25)$$

as elements in $L_{r'_1}(\mathcal{M}, \Omega)^*$. Consequently,

$$\Delta_{\phi_2, \Omega}^{1/p_2} \Delta_{\phi_1, \Omega}^{1/p_1} \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Delta^{1/2p_1}, \quad (26)$$

as elements in $\mathcal{L}_{r'_1, 0}^*(\mathcal{M}, \Omega)$. Iteration of this procedure results in

$$\Delta_{\phi_n, \Omega}^{1/p_n} \dots \Delta_{\phi_1, \Omega}^{1/p_1} \Omega \equiv \Delta^{1/2p_n} A_n \Delta^{1/2p_n} \dots \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Omega \quad (27)$$

as elements in $L_2(\mathcal{M}, \Omega)^*$, because of $\sum_{j=1}^n 1/p_j = 1/2$. But since $\mathcal{H} = \mathcal{H}^* = L_2(\mathcal{M}, \Omega)^*$ the proof is finished. \square

Lemma 4.2. *Let $p \in \mathbb{N}$ be even and $A \in \mathcal{M}^+$. Then there exists $\phi \in \mathcal{M}_*^+$ such that*

$$\|\Delta^{1/2p} A \Omega\|_p = \phi(\mathbb{1})^{1/p} = \omega_\beta(A \Delta^{1/p} A \dots \Delta^{1/p} A)^{1/p}. \quad (28)$$

On the r.h.s. we have used Araki's symbolic notation introduced in the sentence following Equ. (2).

Proof. As proved in Lemma 4.1, there exists $\phi \in \mathcal{M}_*^+$, such that $\|\Delta^{1/2p} A \Omega\|_p^p = \phi(\mathbb{1})$, and $\Delta^{1/2p} A \Delta^{1/2p} \equiv \Delta_{\phi, \Omega}^{1/p}$ as elements in $\mathcal{L}_{p, 0}^*(\mathcal{M}, \Omega)$. Thus, by (19) and (8),

$$\begin{aligned} \omega_\beta(\Delta^{1/2p} A \Delta^{1/2p} \dots \Delta^{1/2p} A \Delta^{1/2p}) &= (\Delta_{\phi, \Omega}^{1/p} \dots \Delta_{\phi, \Omega}^{1/p} \Omega, \Delta_{\phi, \Omega}^{1/p} \dots \Delta_{\phi, \Omega}^{1/p} \Omega) \\ &\leq \phi(\mathbb{1}) = \|\Delta^{1/2p} A \Omega\|_p^p. \end{aligned} \quad (29)$$

Since $\phi \in \mathcal{M}_*^+$, there exists [5] a vector $\xi \in \mathcal{P}^\sharp$ such that $\phi(X) = (\xi, X\xi)$ for $X \in \mathcal{M}$. Using $\xi = J_{\phi, \Omega} \Delta_{\phi, \Omega}^{1/2} \Omega = J_{\xi, \Omega} \Delta_{\xi, \Omega}^{1/2} \Omega$, there holds

$$\phi(X) = (\xi, X\xi) = (\Delta_{\phi, \Omega}^{1/2} \Omega, J_{\phi, \Omega}^* J_{\phi, \Omega} \Delta_{\phi, \Omega}^{1/2} X^* \Omega),$$

where $J_{\phi,\Omega}^* J_{\phi,\Omega} = s_{\mathcal{M}}(\xi) s_{\mathcal{M}'}(\Omega)$ is a projection [3, p. 396]. Therefore

$$\phi(\mathbb{1}) \leq (\Delta_{\phi,\Omega}^{1/2} \Omega, \Delta_{\phi,\Omega}^{1/2} \Omega) = \omega_{\beta}(A \Delta^{1/p} A \cdots \Delta^{1/p} A) ,$$

which finishes the proof. \square

Proof of Theorem 2.2. Assuming the requirements of Theorem 2.2, Lemma 4.1 together with inequality (8), relation (28) and $w_j = z_j - (2p_j)^{-1} - (2p_{j-1})^{-1}$ imply

$$\begin{aligned} \left| \omega_{\beta}(A_n \Delta^{z_n} \cdots A_1 \Delta^{z_1} A_0) \right| &= \left| \omega_{\beta}(\Delta^{1/2p_n} A_n \Delta^{1/2p_n} \Delta^{w_n} \cdots \Delta^{1/2p_0} A_0 \Delta^{1/2p_0}) \right| \\ &= \left| \omega_{\beta}(\Delta_{\phi_n,\Omega}^{1/p_n} \Delta^{w_n} \cdots \Delta_{\phi_1,\Omega}^{1/p_1} \Delta^{w_1} \Delta_{\phi_0,\Omega}^{1/p_0}) \right| \\ &\leq \omega_{\beta}(\mathbb{1})^{1 - \sum_{j=0}^n (p_j)^{-1}} \prod_{j=0}^n \phi_j(\mathbb{1})^{1/p_j} = \prod_{j=0}^n \|A_j\|_{p_j} . \end{aligned}$$

Again we have used Araki's symbolic notation introduced in the sentence following Equ. (2). \square

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